

The inverse problem of Doppler-Zeeman imaging of magnetic CP stars: mathematical model and method of solution

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Abstract. We present the formulation and the development of the inverse problem solution method which permits one to determine surface chemical anomalies and magnetic field configuration in rotating CP stars. The problem is referenced as an ill-posed one. Observed Stokes parameters of absorption lines in stellar spectra are used as input information.

The proposed mathematic model leads to a system of nonlinear integral equations for determining local abundances and local magnetic field vectors. Analytic approximations are used to describe local Stokes profiles and magnetic field configuration. The regularized iteration Newton algorithm is used to solve the integral equation system. The I,V,U,Q-inverting code was tested by numerical model computations and the question of uniqueness of the solution is studied.

1. Introduction

The stars we investigate belong to the type of magnetic chemically peculiar A and B stars which possess large-scale strong surface magnetic fields and great atmospheric chemical anomalies inhomogeneously distributed over their surfaces. To clear up the physical nature of these strange peculiar objects, it is necessary to know the configuration and the value of magnetic fields and surface distribution of chemical elements in their atmospheres.

The only source of such a knowledge is an analysis of spectroscopic and polarimetric observational data on absorption lines in their spectra. Such an analysis requires: 1) formulation of a mathematical model of line profile formation in the atmosphere of a rotating star in the presence of a magnetic field based on the physical theory of spectral line formation in a stellar atmosphere and 2) development of a method of solution of the inverse problem to reconstruct the local Stokes parameters and to transform them into local abundances and local magnetic field vectors.

The first attempt to solve this problem was made by Deutsch (1970) who used spherical functions and Fourier analysis to describe the observed changes of equivalent widths and mean longitudinal component of the magnetic field with period of star rotation. But these input data did not permit use of all information contained in the observed Stokes profiles. This is why Deutsche method could not be used later.

The formulation of the inverse problem in terms of a system of nonlinear integral equations (Khokhlova, 1976, 1986) made it possible to develop the method of solution which uses all information contained in Stokes profiles as input information (Goncharkii et al., 1977, 1982; Vasil'chenko et al., 1996). This method was widely used later under the name of Doppler-Zeeman imaging (Vogt et al., 1987; Brown et al., 1991; Piskunov and Rice, 1993; Khokhlova et al., 1997, 2000).

Below we describe in more detail the points which have been omitted in our previous publications because of lack of space but are useful for a better understanding, and also remove discrepancies in notation and terminology.

2. Mathematical formulation of the problem

According to the theory of spectral line formation in the atmosphere of a rotating star in the presence of a magnetic field, the observed integrated line profile at each moment is the sum of local profiles over the visible star hemisphere. Local profiles depending on the coordinates on an inhomogeneous star surface and in the presence of a magnetic field are the result of polarized light radiative transfer at each point of the star surface and may be computed by solving the radiation transfer equation.

It is known that the properties of polarized radiation can be completely described by four Stokes parameters:

I - overall (full) intensity at a given wavelength,

V - percentage of circular polarized radiation,

U and Q - percentage of linearly polarized light (in some specially chosen orthogonal directions, usually connected with the orientation of the optical axis of linear polarization analyzer).

These values depend on the wavelength inside an absorption line profile and no polarization exists in adjusting continuous spectrum unless the magnetic field exceeds some 10^5 kG. To observe all four Stokes parameters in absorption lines of stellar spectra one must use analyzers of polarized light which in principle are similar to those used for studying sunspot magnetic fields as described, for example, in the book by Bray et al. (1964).

In the case of a star the integral equations describing the observed Stokes parameters are:

$$\bar{I}, \bar{V}, \bar{U}, \bar{Q}(\lambda, \omega t) = \iint_{\cos \theta > 0} I, V, U, Q(M, \omega t, \lambda + \Delta \lambda_D, \vec{H}) \cos(\theta) dM. \quad (1-4)$$

The observed phase dependent Stokes parameters are on the left side of equations, and in the integrand on the right side are local profiles that are dependent on the local abundances, the Doppler shift $\Delta \lambda_D$ and the magnetic field \vec{H} at a point M of the surface.

The Stokes parameters normalized to a nonpolarized continuum are usually measured and they are called Stokes parameter profiles. Taking into account that $R(M, 0) = 1 - I/IC(M, 0)$ and assuming that the specific intensity of the continuum does not depend on coordinates but on angle θ only, one obtains for the polarization profiles the following equations:

$$\frac{1}{\bar{N}} \iint_{\cos \theta > 0} \bar{R}_{I,V,U,Q}(M, \omega t, \lambda + \Delta \lambda_D, \vec{H}) \cos(\theta) dM, \quad (5-8)$$

where $\bar{N} = \iint_{\cos \theta > 0} u_1(\theta) \cos(\theta) dM$ is the normalizing factor, $u_1(\theta)$ is the center-to-limb continuum variation law. For the purpose of normalizing we assume that $Ic(\theta) = 1$, and $u^l(0)$ in the integrand of (5-8) one may consider as the weighting factor.

The system of equations is to be solved numerically by a properly chosen iteration method and for this all functions should be written explicitly, so the mathematical model should be formulated in details. The system (5-8) of four integral equations does permit determination of the local profiles of four Stokes parameters which in turn are determined by four scalars: local abundance and three coordinates of the local magnetic field vector.

The success in solving the problem is greatly dependent on the proper choice of mathematical model. Firstly it must provide an adequate description of the

processes inside the atmosphere of a star, but on the other hand it must be simple enough to permit developing an efficient numerical algorithm for solving the inverse problem.

In present publication we consider the details of physical substantiation of the mathematical model we used and description of our method of solution for the problem of Doppler-Zeeman mapping.

3. Mathematical model

3.1. Description of local Stokes parameter profiles in a star atmosphere

To compute local Stokes parameter profiles which stand in the integrand on the right side of equations (5-8) one should write down the solution of transfer equation for each point M on the star surface for each rotation phase at which a spectrum was taken and all this should be done for each step of an iterative process. The method of numerical solution of the transfer equation is well elaborated now, but it requires too much computer time being repeated many times. This is why we started to use finite-dimensional approximation functions to present the local profiles of the Stokes I parameter at the very beginning of our work (Goncharskii et al., 1977, 1982).

It turned out to be convenient to present all four Stokes parameters by the analytical solutions of transfer equations for polarized light obtained by Unno (1956) and complemented by consideration of the magneto-optical effect in a stellar atmosphere by Rachkovskii (1962), Landolfi and Landi Degl'Innocenti (1982) resumed in the paper by Jefferies et al. (1989).

Our calculations have shown that for early-type stars the Faraday effect is negligible, and then the solution appears to be:

$$R_I(M, \omega t, \lambda + \Delta \lambda_D, \vec{H}) = \frac{\beta \mu}{1 + \beta \mu} \left(1 - \frac{1 + \eta_I}{(1 + \eta_I^2 - \eta_V^2 - \eta_U^2 - \eta_Q^2)} \right), \quad (9)$$

$$R_{V,U,Q}(M, \omega t, \lambda + \Delta \lambda_D, \vec{H}) = \frac{\beta \mu}{1 + \beta \mu} \left(\frac{\eta_{V,U,Q}}{(1 + \eta_I^2 - \eta_V^2 - \eta_U^2 - \eta_Q^2)} \right). \quad (10-12)$$

These analytical solutions were obtained for a simplified line formation model under the assumption of depth-independent ratio of selective to continuum absorption coefficients $\eta = k/k_C$ and also linear depth dependence of source function in continuum $B(\tau) = B_0(1 + \beta \tau)$.

Let us consider first a simpler case of mapping abundance anomalies where the magnetic field is small or the line used has a small Lande factor, so

the effect of magnetic field is negligible. Then the solution of only equation (9) permits one to obtain a map of chemical anomalies (Doppler mapping).

In this case all terms which account for the magnetic field in equation (9) turn to be zero. Remembering that $R_I(\lambda, 0) = (I_C(0) - I(\lambda, 0)) / I_C(0)$ and after simple transformations one obtains for I Stokes parameter profile:

$$R_I(0) = \frac{\beta\mu}{1 + \beta\mu} \frac{\eta_I}{1 + \eta_I}, \quad (13)$$

where $\mu = \cos(\theta)$.

Note that expression (13) resembles by its structure the empirical Minnaert (1935) formula used in our earlier work (Goncharskii et al., 1977, 1982):

$$R_I(\lambda) = R_C \frac{\tau H(a, v)}{R_C + \tau H(a, v)}. \quad (14)$$

This formula was proposed by Minnaert to approximate line profiles in a solar spectrum, and he measured R_c (the central line depth) directly from this spectrum. By the definition $0 < R_c < 1$.

Comparing (13) and (14) one may suggest that in the case of (13) the first factor also plays the role of the line central depth which depends (nonlinearly) on the number of absorbing atoms. We have found it convenient to use as an approximating function the expression:

$$R_I(\lambda) = R_C \frac{\eta_0 H(a, v)}{1 + \eta_0 H(a, v)}. \quad (15)$$

It is possible to choose an approximating function expressing the dependence of R_c on η_0 :

$$R_C = C_1(1 - e^{C_2\eta_0}), \quad (16)$$

where C_1 is the central depth of a very saturated line and C_2 is the properly chosen numerical parameter.

These analytical expressions describe well the dependence of the line profile on the abundance of an element, as η_0 is proportional to the number of line forming absorbing atoms. This permits η_0 to be considered as one of the principal values to be determined when solving the inverse problem.

Expression (13), nevertheless, cannot describe rigorously the center-to-limb variation of a local line profile because the above analytical solution takes into account only the angle θ for the intensity in the continuum. Besides the assumption that the source function is linearly dependent on depth may be wrong in the upper atmospheric layers where strong lines form. This may cause difficulties and errors when estimating local abundances using η_0 found from (13). So one naturally needs to connect profiles (13) with the profiles obtained by numerical integration of the transfer equation for a more adequate atmosphere model.

Numerous observational data show that the atmospheres of CP stars with moderate magnetic fields do

not differ much from the atmospheres of normal main sequence stars, and well elaborated theoretical atmosphere models such as Kurucz (1992) models with appropriate parameters T_{eff} and $\log g$ may be used in this case.

The most explicit way of using theoretical profiles for Doppler imaging was taken by Hatzes (1991) and also by Piskunov and Rice (1993) who precalculated and stored in memory a grid of profiles as a function of abundance and in the process of iterations retrieved and interpolated data from tables.

In our method we also precalculate by numerical integration a grid of Stokes specific intensity profiles R_c for a set of abundances for each line we use for mapping and we approximate these profiles by formula (15). The special code has been developed to choose parameters of the approximating function which was described in details in section 2.2.1 of our paper (Khokhlova et al., 1997).

These parameters, except η_0 , do not depend on coordinates on the star surface. Practically they do not depend on the value of η_0 (the line intensity) either. The dependence of the line profile on θ is taken into account by the factor $U_2(\theta)$. This function presents a linear or quadratic expansion by $\mu = \cos(\theta)$, coefficients being found from the set of theoretical profiles for each particular line. The line may become stronger or weaker from center to limb, depending on ionization and excitation potentials. This technique is also demonstrated in Fig. 2 in the paper by Khokhlova et al. (1997).

It is clear that when the role of a magnetic field cannot be neglected, and it is necessary to solve simultaneously the four equations (5-8), the problem is getting much more complicated. In this case the local polarization profiles depend not only on the local value of the magnetic vector at point M but also on an instantaneous value of the angle between this vector and the line of sight at point M , which changes during the star rotation. In this case the precalculation of the local profiles, retrieval and interpolation from tables in the process of iterations become unrealistic even for big computers. Application of analytical approximations is practically the only way to solve the problem.

To describe the Stokes profiles in the case of magnetic field, we use analytical solutions (9-12). In the quantities $\eta_{I,V,U,Q} = \eta_0 f_{I,V,U,Q}(\lambda, \vec{H})$ the functions $f(\lambda, \vec{H})$ are convolutions of each component of the Zeeman pattern split by the local magnetic field at the moment of phase ωt with a Voigt profile (the group of π components as well as the right- and left-polarized σ components are treated separately). The Zeeman patterns and relative intensities of components are known from the classical physics (for example, see Condon and Shortley, 1951), and for LS

coupling they were calculated by Beckers (1969). All formulae we used for this general case are given in our papers (Vasil'chenko et al., 1996 and Khokhlova et al., 1997).

In conclusion of this section we note some statements which justify our use of analytical approximations of the local Stokes profiles:

1. The transfer of light of one of polarization states (σ +, π , or σ — component) may be considered independently of each other.

2. The intensity profile of each polarized Zeeman component is formed in the same way as the profile of intensity of unpolarized light.

3. The difference between Miln-Eddington atmosphere and a more sophisticated modern computed star atmosphere is more important for parameter η_0 and hence the intensity profile. But well chosen parameters of approximation described by Khokhlova et al. (1997) make the difference between intensity profiles rather small. We may refer also to the results reported by Hardorp (1976), which show that the numerical solution of the transfer equation for a magnetically splitted profile and analytical formula of Unno type lead to similar results.

4. The assumption that the atmosphere model is independent of coordinates is doubtful when the magnetic field and chemical patches on the star surface are strong. One cannot be sure that the local profiles calculated by numerical solution of the transfer equation for one fixed atmosphere model are valid for the whole star, no matter how they are used: by extraction from tables or by analytical approximation.

5. We have found that the magnetic field configuration obtained from lines with different Zeeman patterns is practically the same. This is an evidence that no gross errors arise due to our analytical approximation of local Stokes profiles.

3.2. Analytical description of magnetic field configuration

All measurements of effective magnetic fields of CP stars that have been made up to now show that most of them have large-scale regular dipolar magnetic field structures, but now a few stars are known with a quadrupole component. For example, in the B2V He-variable CP star HD 37776 the quadrupole component is dominant, and even an octupole component was suspected (Thompson and Landstreet, 1985; Khokhlova et al., 2000)

This makes it natural to search magnetic configuration as an expansion of spherical harmonics of magnetic multipole potentials to an arbitrary high order (Bagnulo et al., 1996). In principle, this permits one to describe any field configuration, but taking into account higher number of multipoles one gets instability of the solution due to incompleteness or inac-

curacy of input data. It was shown (Khokhlova et al., 2000) that in the case of HD 37776 having a dominant quadrupole field, the addition of octupole-produced instability (in the sense that octupole vectors derived from different spectral lines spread over a big area). This question is considered in more details in section 5 when discussing the problem of uniqueness of the solution.

4. Method of solution

From the mathematical point of view the mapping problem leads to the system of integral equations:

$$R_I(\lambda, \omega t) = \frac{1}{N} \iint_{\cos \theta > 0} u_1 u_2 R_C(\eta_0) \times \left(1 - \frac{1 + \eta_I}{(1 + \eta_I^2 - \eta_V^2 - \eta_U^2 - \eta_Q^2)} \right) d\sigma, \quad (17)$$

$$R_{V,U,Q}(\lambda, \omega t) = \frac{1}{N} \iint_{\cos \theta > 0} u_1 u_2 R_C(\eta_0) \times \left(\frac{\eta_{V,U,Q}}{(1 + \eta_I^2 - \eta_V^2 - \eta_U^2 - \eta_Q^2)} \right) d\sigma. \quad (18-20)$$

Here the unknowns are functions of two arguments — function of distribution of chemical composition $\eta_0(L, \varphi)$ and distribution of magnetic held $\vec{H}(L, \varphi)$ over the star surface. The functions $\eta_{I,V,U,Q}$ in equations (17-20) can be written in the form $\eta_{I,V,U,Q}(\lambda, \vec{H}) = \eta_0 f_{I,V,U,Q}(\lambda, \vec{H})$, where the functions $f_{I,V,U,Q}(\lambda, \vec{H})$ are denned in (3.1) and represent non-linear functions of their arguments. Thus the system of integral equations (17-20) is non-linear.

In this method we use parametric representation of the magnetic field on the star surface which may be either a displaced dipole model or decomposition to axially symmetric spherical harmonics up to the third order (dipole, quadrupole and octupole moments). Including of higher orders in our code is also possible. Let us denote the set of parameters that define magnetic field as $\mathbf{h} \in \mathbf{R}^*$ when using the displaced dipole model and as $\mathbf{h} \in \mathbf{R}'$ for spherical harmonic decomposition.

Designating integral operators in (17-20) as $F_z(\mathbf{h}, \eta_0)$, we write it in a more compact form:

$$R_z = F_z(\mathbf{h}, \eta_0), \quad \mathbf{x} = \{I, V, U, Q\}. \quad (21)$$

We will suppose that all the observed profiles $R_z(\lambda, \omega t)$ belong to $L_2[\Pi]$, where $\Pi = \{\lambda \in (\lambda - \delta\lambda, \lambda + \delta\lambda), \omega t \in [0, 2\pi]\}$. The value $\delta\lambda$ is defined as half of the full width of the spectral line under investigation. The choice of the space L , is determined by the usage of mean square metric to measure the discrepancy between the observed and synthesized profiles. To solve the inverse problem, we have also to

determine a space the solution will belong to. Let the function η_0 be of space T , and the set of unknowns $z = (h, \tau) \in Z = R^r \times T$ ($Z = R^r \times T$). To choose space T one has to take into account the following aspects:

- operators F_x are to be defined in Z or its closed subset;
- convergence in space T should guarantee a desired convergence of the approximate solutions;
- the effective algorithm of the solution of the inverse problem for non-linear integral equations in space T exists.

Here we use the Hilbert space $L_2(\Omega)$ as T , where Ω is the surface of the unit sphere. This sphere can be parameterized, for instance, by longitude φ and latitude L . The main reason for this choice is the simplicity of the numerical implementation of the algorithm. It is easy to see that integral operators F_x are continuous and even completely continuous in the pair of spaces $Z \rightarrow L_2(\Pi)$. Unfortunately, use of space $T = L_2(\Omega)$ does not guarantee uniform or even pointwise convergence of approximate solutions. To force stronger convergence one needs to use stronger metrics in space T , for example $T = W^1(\Omega)$ while developing methods of solution for the inverse problem.

At the present time only wide-band data of linear polarization measurements (U and Q Stokes parameters) are available, and there are only a few stars that provide sufficient data on circular polarization. Therefore we need sometimes to solve the problem of finding chemical composition and magnetic field of a star by an incomplete set of input Stokes parameters.

The following cases of indices X may be realistic to describe different statements of the problem when different observational data of Stokes parameters are available:

- only information for non-polarized spectra is available;
- both intensity and circular polarization profiles are available; for now this is the most usual case;
- all Stokes line profiles are available - this is the most favourable case to solve the inverse problem. But measurement of the U and Q Stokes line profiles is a rather difficult technical problem and the set of these data may be replaced by integrated broad-band linear polarization data.

Our experiments showed that if a star has a sufficiently strong magnetic field, so that magnetic splitting of Zeeman components is greater than rotational Doppler widths, one can find chemical composition and magnetic field even when only I Stokes parameter is available (the case of Babcock star, Khokhlova et al., 1997). To solve the mapping problem for stars with relatively weak magnetic fields one needs data

on circular polarization in addition (numerical experiments).

The properties of the operators $F_x : Z \rightarrow L_2(\Pi)$ make us use special algorithms to solve the minimization problem which guarantee stability of the obtained approximate solutions (Tikhonov et al., 1995). Regularized methods to minimize the discrepancy functional

$$\Phi(z) = \sum_{x \in X} \|F_x(z) - R_x\|^2$$

are used to solve non-linear ill-posed problems.

In the case $T = L_2(\Omega)$ one of the most effective methods of solving the system of non-linear integral equations (21) is the Newton's iterative method as described for example by Bakushinskii and Goncharskii (1994). Given the current approximation z_k , the next one can be calculated using the formula

$$z_{k+1} = z_k + \left\langle \sum_{x \in X} F_x'^*(z_k) F_x'(z_k) + \alpha_k E \right\rangle^{-1} \times \left[\sum_{x \in X} F_x'^*(z_k) (F_x(z_k) - R_x) + \alpha_k z_k \right]. \quad (22)$$

Here $F_x'(z)$ is the derivative of the operator F_x at point z , and $F_x'^* : L_2(\Pi) \rightarrow Z$ is the operator conjugate to $F_x'(z)$, α_k is the sequence of positive numbers tending to zero. Operators $\left\langle \sum_{x \in X} F_x'^*(z_k) F_x'(z_k) + \alpha_k E \right\rangle^{-1}$ exist and are continuous owing to the positive value of α_k . To guarantee the stability of the approximate solutions, the sequence α_k must not decrease too fast. Generally speaking, one has to choose α_k basing on the investigations of the properties of the non-linear operators F_x , but it is well known that $\alpha_k \approx 1/\sqrt{k}$ usually provide convergence and stability of the approximate solutions if the first approximation is chosen sufficiently close to the exact solution. We used the sequence

$$\alpha_k = \alpha_0/\sqrt{k}, \quad (23)$$

where the value of α_0 was found by numerical experiments.

Iterative algorithms, used to solve ill-posed problems, are to be supplied by the so-called stopping rule. This implies that one has to take as the approximate solution of the ill-posed problem the iteration of (22) with the number k , that corresponds to the observational data precision. The more precise is the input information, the greater number of iterations must be calculated by (22) to get a stable approximate solution.

The method which uses (22), (23) guarantees the stability of the approximate solution if we use a stopping rule like $n \approx 1/\sqrt{(\delta)}$, where δ is the precision

of the observational data in mean-square metric. We used

$$n = n_0/\sqrt{\delta} \quad (24)$$

and adjusted the value of n , empirically.

As it was mentioned above the method using (22-24) gives a stable solution only if the first approximation z_0 is chosen to be close to the unknown exact solution. Unfortunately, in practice it is impossible to verify conditions imposed on the first approximation by Bakushinskii and Goncharskii (1994) due to the too complex form of the operators $F_x(z)$. Numerical experiments show that the direct application of the method (22-24) could be successful only if a very good first approximation is available.

To solve integral equations (21) in the case where the quality of the first approximation z_0 cannot be a priori estimated, we used the regularized iterative method in the form:

$$z_{k+1} = z_k + \left\langle \sum_{x \in X} F_x'^*(z_k) F_x'(z_k) + \alpha E \right\rangle^{-1} \times \left[\sum_{x \in X} F_x'^*(z_k) (F_x(z_k) - R_x) + \alpha z_k \right] \quad (25)$$

with the constant regularization parameter α . This is just the Newton's method to minimize smoothing functional as described by Tikhonov et al. (1995). The regularization parameter was chosen based on numerical experiments.

We used discrepancy method as a stopping rule. According to this method one has to continue iterations (22) or (25) until the discrepancy reaches the value of the precision of input information, that is

$$\Phi(z_k) > \delta, \quad \text{and} \quad \Phi(z_{k+1}) \leq \delta. \quad (26)$$

This method has proved to be effective while solving a lot of applied problems. Non-adhering to the stopping rule and the use of too many iterations leads to instability of the obtained solutions and therefore to false decisions on the structures of the surface distributions in the star atmosphere.

Note that the possibility of choosing the number of iterations from condition (26) depends on the adequacy of the used mathematical model and in turn can serve to prove this adequacy.

We have performed a lot of numerical experiments and have investigated the properties of methods (22), (23), (26) and (25), (26) for different model distributions. It appears that the first combination of (22), (23), (26) allows one to investigate very detailed distributions, but requires a good first approximation. The second combination of (25), (26) is not sensible to the first approximation, but makes convergence too slow. It also became clear that the convergence of the approximate solutions in the metric of the space L_2

is not sufficient enough. Hence, the following modification has been developed for further usage.

It was decided to use the smoothing functional in the following form:

$$M(z) = \Phi(z) + \alpha \|\nabla(\eta_0)\|^2 = \sum_{x \in X} \|F_x(x) - R_x\|^2 + \alpha \|\nabla \eta_0\|^2. \quad (27)$$

The additional term with the gradient of unknown function η_0 does not allow large oscillations of the local abundance when minimizing $M(z)$. As well it smoothes the edges of abundance distribution. Regularization of the zero order in the form of (25) may lead to a more "detailed" distribution especially if the input information error level is underestimated. But these details usually have artificial or computational origin resultant from the ill-posed nature of the problem and cannot be regarded as real ones. We do believe that one has to find as much smooth solutions of (21) as possible.

In this case the regularized Newton's method can be written as

$$z_{k+1} = z_k + \left\langle \sum_{x \in X} F_x'^*(z_k) F_x'(z_k) + \alpha_k \nabla^* \nabla \right\rangle^{-1} \times \left[\sum_{x \in X} F_x'^*(z_k) (F_x(z_k) - R_x) + \alpha_k \nabla^* \eta_{0k} \right]. \quad (28)$$

The following expression for the sequence α_k was used (α_0 being chosen empirically):

$$\alpha_k = \frac{\alpha_0 \|\Phi_k\|^2}{\|\nabla \eta_{0k}\|^2}. \quad (29)$$

This sequence (29) assures that the regularization parameter decreases as the approximation tends to the exact solution. We used a stopping rule in the form (26).

We used the following finite-dimension approximation of the data to implement the numerical method of solving (21). The function $\eta_0(L, \varphi)$ was approximated as a piecewise constant function on the rectangular grid on the visible surface of a star and was represented as the vector $\vec{\eta}_0$ of dimension $n = n_l \times n_\varphi$. Input data — Stokes line profiles — were given for the set of rotation phases ωt_i and each of these profiles was represented as a set of values at the wavelengths in the vicinity of the central wavelength λ_0 . Thus, the input data are also the finite-dimension vector \vec{u} . Now we can approximate non-linear operators F_x as functions of the finite number of arguments $n + 6$. The derivatives of the operators are approximated as the matrix of partial derivatives and the conjugate operator can easily be found as conjugation of matrices.

Table 1: *Models of magnetic field which were tested*

	mod1			mod2		
	L	φ	H_p	L	φ	H_p
D	0	45	10	0	0	10
Q	45	45	10	45	-45	10
Oct	90	45	5	-	-	-

As we have already mentioned, the matrix approximating $\sum_{z \in X} F_x'(z_k) F_x'(z_k) + \alpha_k E$ is well-conditioned due to the $\alpha_k > 0$ and we can use standard algorithms of linear algebra to inverse the matrix when solving the system of linear equations.

5. On uniqueness of the solution and testing of the method by model computations

The question on uniqueness of the solution obtained by Doppler-Zeeman imaging is obviously of a great importance and it should be thoroughly studied. The answer will certainly depend on the precision and completeness of the observed input information.

Using analytical expressions for the magnetic field as expansion in series of spherical harmonics of magnetic multipoles potentials may lead to a situation, when various sets of multipoles create a magnetic field configuration which is similar to the real one, thus providing several local minima of the residual. It is important for us in such a case to know that these configurations are really very close to each other. In such a case one may speak not about ambiguous solution but rather about ambiguous mathematical description of the same true solution.

The most convenient way to study the problem of uniqueness of the solution, as it was mentioned earlier, is numerical testing of various models. We have found by such testing that non-uniqueness appeared depending on the model which is used as an initial approximation: if it is taken to be not too far from a true model, the solution converges to the true model. It is convenient to take as such initial approximation the homogeneous abundance distribution equal to the observed averaged over the star disk abundance and roughly visually guess estimation from the observed V profiles of the position and strength of magnetic poles. If the initial approximation is taken arbitrary to be far from the true model, then the minimization of the residual may lead to false minima.

Below are some examples of such situations. The models which were tested are shown in Table 1. Magnetic fields were formed by combinations of non-axial but centered dipole, quadrupole and octupole with coordinates L , φ and polar strength H_p (kG).

The testing was made for the angle $i = 45^\circ$ and $V_{\text{rot}} \sin i = 50 \text{ km/s}$. The "observed" profiles of the

Table 2: *Starting models used as initial approximation for iterations*

	str1			str2		
	L	φ	H_p	L	φ	H_p
D	60	60	15	60	60	15
Q	60	60	15	105	0	5
Oct	60	60	15	0	-30	2

Table 3: *Model solutions obtained*

No.		L	φ	H_p	σ	min
1	D	0	45	10	0.00008	I,V,U,Q
	mod1+m2a	45	45	10		
	str1+m300	90	45	5		
2	D	0	45	10	0.00022	I,V
	mod1+m2a	45	45	10		
	str1+m100	90	45	5		
3	D	-7	43	10	0.0006	I,V
	mod1+m2a	55	44	12		
	str2+m100	-37	17	2		
4	D	5	46	15	0.0031	I,V,U,Q
	mod1+m2c	108	-42	-6		
	str2+m100	-99	24	4		
5	D	22	38	12	0.0013	I,V,U,Q
	mod1+m2a	128	31	-5		
	str1+m300	-103	-55	-13		
		(77	55	13)		
6	D	0	0	10	0.00007	I,V
	mod3+m2b	45	-45	10		
	str2+m100	-	-	-		
7	D	-1	2	9.5	0.00046	I,V
	mod2+m2c	45	-44	9.3		
	str2+m100	-	-	-		

Si III line 4574 \AA (normal Zeeman triplet) were computed for the atmosphere of a B2V star, similar to HD 37776, studied by us (Khokhlova et al., 2000). Model Stokes profiles were computed for three kinds of abundance distribution:

- a) a spot with ten-fold underabundance (m2a),
- b) a spot with ten-fold overabundance (m2b),
- c) a highly overabundant strip located along the position of the maximum tangential magnetic field component (m2c).

The computed model profiles were used as input information to solve the model inverse problem. In Table 2 the models used as initial approximations are shown, and the solutions obtained are given in Table 3.

In the first column of Table 3 the upper row indicates the test number, the next row is the model tested from Table 1, and the lower row indicates the initial approximation model for the magnetic field and abundance distribution.

In the second and third columns solutions (multi-

poles and their coordinates) are given.

In the fourth column are the residuals and in the fifth is the type of solution (I,V,U,Q-inversion or I,V-inversion)

Figures 1-7 demonstrate the original abundance and the magnetic field configuration together with those reconstructed by the inversion procedure. The examples No.1 and No.2 show that if the initial approximations are chosen properly, the solutions converge to the original models precisely even without linear polarization input data. The minimization of the residuals was made by the I and V Stokes profiles only in that case, but the U and Q profiles were computed and compared as well. The minimization by I and V was enough to decrease residuals for U and Q.

In the examples No.3 - No.5 the initial approximations are farther from the original model, or the model is more complex. In these cases residuals are about an order of magnitude larger than in the cases No.1 and No.2. It is important to note that they still remain smaller than the possible error in the available observed profiles. But what is more important, these solutions describe magnetic field configurations which are close enough to the original "true" models. The examples No.6 and No.7 show that in the absence of an octupole component in the magnetic field configuration it is easier to get a "true" solution.

The testing showed that the difference in the abundance distribution does not influence noticeably the determined magnetic configuration. At least two recommendations follow from the above results of testing, which may help find "the best" solution:

1. If only one spectral line of an element is used for D.-Z. mapping, it is necessary to compare the results obtained with different initial approximations.
2. Several lines of one element should be used for mapping.

Another important conclusion follows from the tests described above: fine details of magnetic configuration of models can be revealed only when the residual is very small — much below observational errors for real stars.

The real error in the observed Stokes parameters is determined not only by the S/N ratio, which now probably may reach a value of 500 (which would coincide with $\sigma = 0.002$), but according to our experience it is limited by a value of > 0.005 because of unrevealed blending by very faint lines and by errors in drawing the continuum.

The necessity of stopping iterations at this value of the residual (see stopping rule (26) in section (4)) may not allow the best of possible solutions to be achieved. Nevertheless our test shows (Figures 1-7) that the solutions obtained give a correct (though inevitably rough) idea of magnetic field configuration and permit studying its connection with the location

of chemical anomalies. The limit is set at the present time mainly by observational errors, and also by a poor knowledge of the local atmosphere structure.

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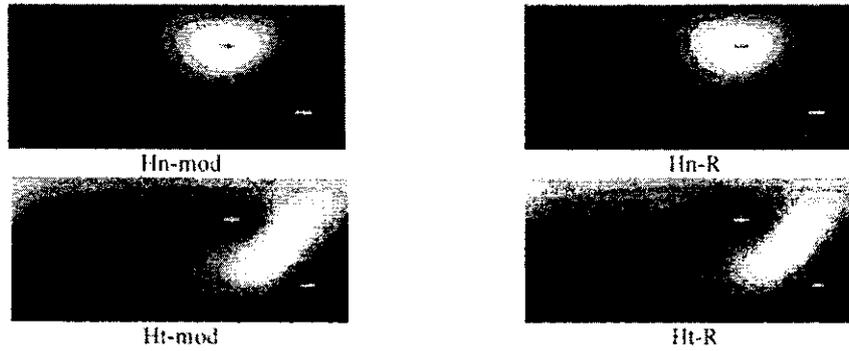


Fig. 1 The maps in rectangular projection of a distribution of normal and tangential magnetic field components over a star surface H_n and H_t for model (mod) and reconstructing solution (R). Increasing of the values correspond to light in the pictures.

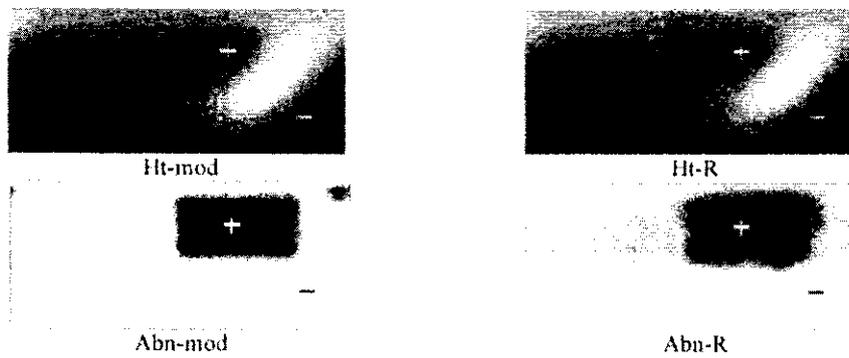


Fig. 2 The same as fig.1 but supplemented with the abundance distribution maps (Abn)

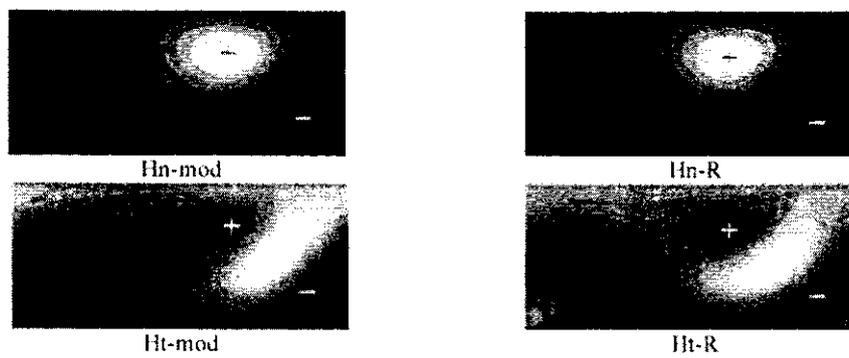


Fig. 3 The same as fig.1 (see text)



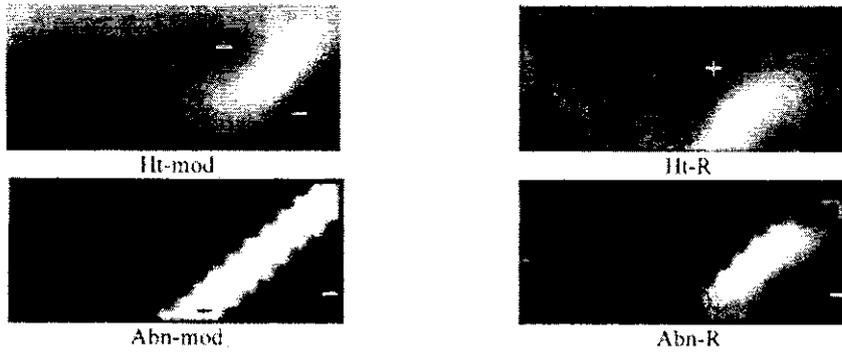


Fig. 4 The same as fig.2 (see text)

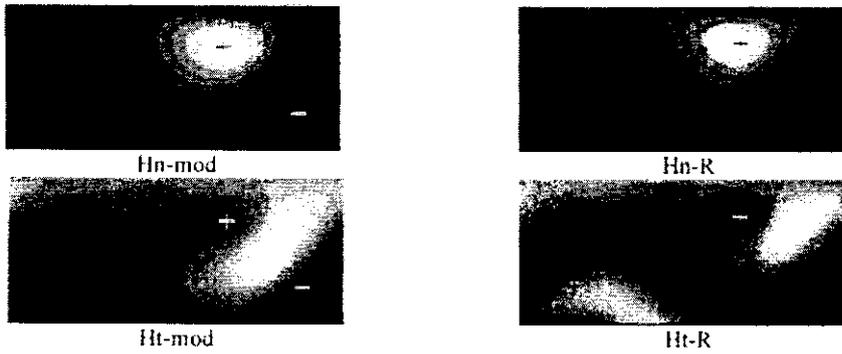


Fig. 5 The same as fig.1 (see text)

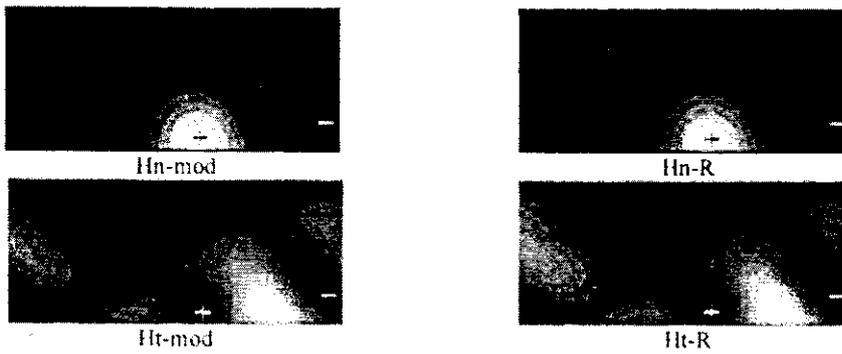


Fig. 6 The same as fig.1 (see text)

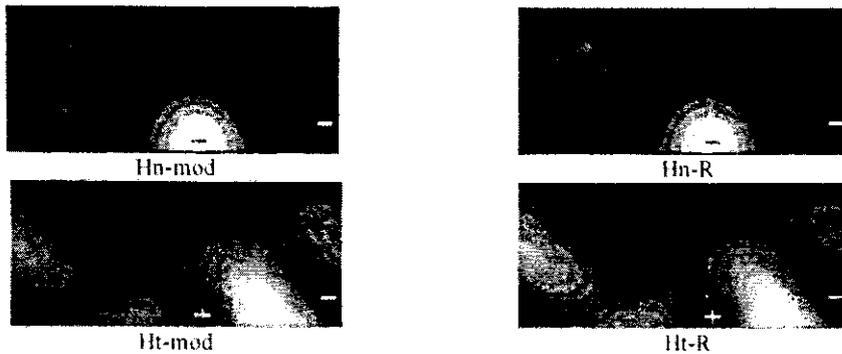


Fig. 7 The same as fig.1 (see text)